Radiant Heat Exchange in a Gas-filled Enclosure: Allowance for Nonuniformity of Gas Temperature

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A method is presented for predicting the effect of allowance for radiation exchange on the distribution of temperature and heat transfer within a furnace chamber. The system is divided into surface zones and gas zones, the number being dependent on the desired accuracy of the result. Direct-exchange factors are available for gas-gas, gas-surface, and surface-surface zone interchange. From these factors one can determine the net exchange factor for any zone pair, making due allowance for interaction with all other zones. The resultant factors are then fed into a set of energy balances, one on each zone, which by simultaneous solution permit a determination of the space distribution of gas and surface temperatures and the distribution of heat flux over the surfaces.

The problem of the performance of an industrial furnace of specified shape and size, fed with fuel and air at specified rates and in a specified pattern, may be said to be solved when the temperature pattern in the gas space and along the walls can be predicted. The problem is in principle of course capable of solution if sufficient knowledge exists concerning the factors which control the flow pattern, the progress of combustion, and the transfer of heat by radiation and convection at every point in the system, but no one expects that an early solution to so extraordinarily complex a problem will be soon forthcoming. Presently available methods (*5*) permit determining the details of surface-temperature variation and surface-heat-flux variation for the special case of an assumed uniform temperature and composition of the gases in the furnace enclosure. The object here is to advance one stage further toward a general solution by removing the restriction of gas-temperature uniformity and to establish, in the course of the solution, space variation both in surface and in gas temperatures. The analysis is limited, however, to solution of the heat transfer problem; i.e., a description of the combustion and mixing patterns in the furnace enclosure is assumed available. Although the analysis is not limited to a system in which all parts of the gas volume are assumed to have identical values of absorption coefficient, allowance for departure from that assumption is only approximate. Despite these limitations, the new method enormously widens the range of problem types capable of solution. Although directed primarily toward application to industrial furnace chambers, the method can be used on such problems as studying temperature distribution in a glass pot.

If the bounding walls of an enclosure and the gas space in it are both divided into zones small enough to be considered isothermal, an energy balance may be made on each zone. For a gas zone the sum of the radiant energy ultimately received from all zones in the system (both gas and surface) plus the net conduction or convection to it from adjacent gas or surface elements plus the net enthalpy flux (chemical and sensible) to it due to bulk flow must equal the radiation originating within the gas zone—plus any increase in enthalpy of the zone during the time interval if the process is unsteady in time. Similarly, for a surface zone the sum of the radiant energy received from all zones (gas and surface) plus the net convection to the surface must equal the sum of the emission from the surface, the net flux out through it, and—if the process is unsteady in time—its increase in enthalpy. For any specific problem for which the flow pattern is given or assumed, every term in these equations, with the exception of the net wall fluxes, may be written as a function of unknown temperatures only, and the number of equations will be found equal to the number of unknowns—temperatures or wall fluxes. A solution of a set of these equations, one for each zone in the system, would yield the desired distribution of temperature and flux throughout the enclosure. The reader will note that when the zones are made infinitesimal the resulting quantitative relation is an integral equation and that the complicated boundary conditions make an analytical solution out of the question. The concept of a finite number of zones must consequently be retained.

It is necessary to evaluate the emission from either a gas zone or a surface zone and the radiant interchange between any two of these zones, making due allowance for absorption along every beam from one zone to another and for partial diffuse reflection of every beam at every surface, *ad infinitum*. Solution of the geometrical portion of this problem independently of the temperature distribution necessitates first a choice of a specific geometrical grid system. The choice was made of gas cubes or shapes constructable from cubes, and surface squares or shapes constructable from squares, with both the cube edge and the square side designated as B. With few exceptions, all enclosures of industrial importance may be adequately approximated as a cube-square system. (Work on the cylinder system is in progress.)

Initially, discussion will be restricted to a gas the absorptivity of which is independent of wave length and temperature, a "gray" gas. (This restriction will be removed later.) It may readily be shown that the emission rate dE_g from a volume dV of gas is given by

$$dE_g = 4K\mu^2\, dV \cdot \sigma T_g^4 \qquad (1)$$

where k is the absorption coefficient or fractional absorption by the gas per unit path length as the latter approaches zero, μ is the refractive index of the medium, and σT_g^4 is the rate of emission, into a medium of refractive index 1, of radiation from the unit area of a black surface or complete radiator at temperature T_g.

Then the emission rate E_g from a cube of a gray gas of edge B is given by

$$E_g = 4(kB)(B^2) \cdot \phi \cdot \mu^2 \sigma T_g^4 \qquad (2)$$

where ϕ is the fraction of the energy originating within the cube which leaves the boundaries of the cube. The escape factor ϕ, which is a function of kB only, has been obtained by multiple graphical integration as described in Appendix 1. A plot of the factor ϕ against kB is shown in Figure 1.

The emission rate from a surface square of side B into the medium of refractive index μ is given by

$$E_s = \epsilon_s(B^2) \cdot \mu^2 \sigma T_s^4 \qquad (3)$$

where ϵ_s is the emissivity of the surface, and all other symbols are as before except that the subscript s indicates that reference is to a surface zone. For enclosures containing gas, μ is nearly 1 and may be ignored.

DIRECT INTERCHANGE BETWEEN ANY TWO ZONES

The interchange between any two zones in the system is the sum of the direct radiant flux between the two zones and the infinite number of reflected beams within the enclosure due to energy originating at either of the zones under consideration. (Note that the other zones in the system enter this formulation, if they are surfaces, only as reflectors and not as original radiators, and if they are gas zones, only as absorbers.) It is profitable to consider first only the direct interchange between two zones, i.e., the interchange in a black-walled system.

The following nomenclature will be established for one-way radiation flux, surface to surface, surface to gas, gas to to surface, and gas to gas, respectively:

$$q_{s_i s_j} = (\overline{s_i s_j})\mu^2\sigma T_{s_i}^{\ 4} \qquad (4)$$

$$q_{s_i g_j} = (\overline{s_i g_j})\mu^2\sigma T_{s_i}^{\ 4} \qquad (5)$$

$$q_{g_i s_j} = (\overline{g_i s_j})\mu^2\sigma T_{g_i}^{\ 4} \qquad (6)$$

$$q_{g_i g_j} = (\overline{g_i g_j})\mu^2\sigma T_{g_i}^{\ 4} \qquad (7)$$

Here q is the one-way radiant heat transfer from the source zone, indicated by the first subscript, to the receiver zone, indicated by the second; the terms $\overline{s_i s_j}$, $\overline{s_i g_j}$, $\overline{g_i s_j}$, and $\overline{g_i g_j}$, are designated as *direct-interchange areas—direct interchange* to remind the reader that a description of both the sender and the receiver of the radiation is essential and *area* to indicate the dimensions of the term.

The direct-interchange area may be thought of as the product of two terms. The first, having the dimensions of area, is the actual area for a surface zone and *(4)* (volume) (absorption coefficient) for a gas zone. The second is the *reception factor* f of one of three kinds. When the receiver is a surface element separated from the emitter by a nonabsorbing gas and the emitter is a surface element, f has been referred to in the literature as the *direct-view factor* F (*5*). For flux from emitter through an absorbing gas to a surface, f is the product of a direct-view factor and a gas transmittance τ, although the two are not strictly separable after integration over a finite receiver. When a gas volume is the receiver, f includes as an additional multiplier the absorptivity α of the receiver (again, $f \equiv F\tau\alpha$ is not separable into F, τ, and α). In general, then, f is the fraction of the energy originating in any zone in the enclosure which reaches and is absorbed by any other zone (surfaces are for this definition black). The two subscripts on f indicate in sequence the emitter and absorber.

Values of these reception factors, f, for the interchange between two squares in either parallel or perpendicular planes, for a square and a cube, and for two cubes have been obtained by multiple graphical integration as described in Appendix 2. A set of recommended working plots, Figures 2 through 5, give the various types of reception factors as functions of $\Delta x/B$, $\Delta y/B$, and $\Delta z/B$—the distances between the centers of two zones measured along the three coordinate axes—and kB, where k is the absorption coefficient of the gas. For parallel squares, Figure 2, the unique dimension z is the distance between their planes; x and y are interchangeable. For perpendicular squares, Figure 3, the unique dimension z is the distance not perpendicular to the plane of either square. For cubes to squares, Figure 4, the unique dimension z is the perpendicular distance from the square plane. For Figure 5 all three dimensions are interchangeable. To condense Figures 4 and 5 it was found that division of f by ϕ (Figure 4) and by $kB\phi$ (Figure 5) was desirable before plotting. It is to be noted that for any gas cube g_i the escape factor ϕ used in correlation is equal to $(1 - f_{g_i g_i})$, or the complement of the self-irradiation factor of the cube.

Figures 2 to 5 give f's for values of the ratio (center-to-center distance between radiator and receiver)/(edge dimension of either) up to at least 3. For greater separating distances, f rapidly approaches a limiting value capable of simple analytical expression. (See Appendix 3.)

Since the net flux between a gas volume and a surface element, equal to the difference of two one-way flux terms, must become zero when the two radiators are at the same temperature, it follows that, if the absorption coefficient is the same in the two one-way terms regardless of temperature, then

$$\left.\begin{aligned} & & \overline{g_1 s_2} &= \overline{s_2 g_1} \\ &\text{or} & 4kV_{g_1}\cdot f_{g_1 s_2} &= A_{s_2} f_{s_2 g_1} \\ &\text{or} & 4kBf_{g_1 s_2} &= f_{s_2 g_1} \end{aligned}\right\} \qquad (8)$$

Similarly,

$$\overline{g_1 g_2} = \overline{g_2 g_1}, \quad \text{and} \quad \overline{s_1 s_2} = \overline{s_2 s_1} \qquad (9)$$

These constitute, for direct radiation exchange, the general statement of reciprocity, of which the older relation, $A_1 F_{12} = A_2 F_{21}$ (or $\overline{12} = \overline{21}$)(*5*), was a specific example.

Since the reception factors have been evaluated for cubes and squares, the enclosure must be so divided into isothermal zones that the latter can be built up from cubical or square units. The value of B to be used is the largest factor common to every zone in the enclosure. The size of the zone is limited by the maximum volume or area which is justifiably considered isothermal and to a lesser extent by the desire to keep the "view" of surrounding zones the same for all the units comprising the zone in question. This second limitation is of lesser importance, since it affects only the distribution of radiation on reflection or reemission following absorption, and not any original beams. The technique of building up the factors for more complicated shapes may be illustrated by reference to Figure 6. The one-way reception of energy at gas zone g_1 consisting of two units, due to emission from the gas contained in gas zone g_1, is

$$\frac{q_{g_1 \to g_1}}{\mu^2\sigma T_{g_1}^{\ 4}} \equiv \overline{g_1 g_1}$$

$$= 4kB^3(f_{g_a g_b} + f_{g_a g_a} + f_{g_b g_a} + f_{g_b g_b})$$

$$= 4(kB)(B^2)(2f_{g_a g_b} + 2(1 - \phi)) \qquad (10)$$

Similarly, the reception at surface zone s_1, consisting of three units, from gas zone g_1 is

$$\frac{q_{g_1 \to s_1}}{\mu^2\sigma T_{g_1}^{\ 4}} \equiv \overline{g_1 s_1}$$

$$= 4(kB)(B^2)(f_{g_a s_c} + f_{g_a s_d} + f_{g_a s_e}$$

$$+ f_{g_b s_c} + f_{g_b s_d} + f_{g_b s_e}) \qquad (11)$$

A corollary of this discussion is that the total of all the "areas" representing flux from any one zone to each of the others in the enclosure, including itself, must equal the energy originating from that zone, per unit emissive power, or

$$\overline{s_1 s_1} + \overline{s_1 s_2} + \overline{s_1 s_3} + \cdots + \overline{s_1 g_1}$$

$$+ \overline{s_1 g_2} + \overline{s_1 g_3} + \cdots = A_{s_1} \qquad (12)$$

or

$$\sum_i \overline{s_1 s_i} + \sum_i \overline{s_1 g_i} = A_{s_1} \qquad (13)$$

and

$$\overline{g_1 g_1} + \overline{g_1 g_2} + \overline{g_1 g_3} + \cdots + \overline{g_1 s_1}$$

$$+ \overline{g_1 s_2} + \overline{g_1 s_3} + \cdots = 4kV_{g_1} \qquad (14)$$

or

$$\sum_i \overline{g_1 g_i} + \sum_i \overline{g_1 s_i} = 4kV_{g_1} \qquad (15)$$

TOTAL INTERCHANGE BETWEEN ANY TWO ZONES: ALLOWANCE FOR REFLECTIONS AT THE WALLS

Discussion so far has been restricted to a black-walled system. Actual furnace walls, however, are partial reflectors and, fortunately for the present purpose, much more nearly diffuse than specular reflectors; i.e., their reflection, like black-surface emission, follows Lambert's cosine law. This reflected energy is distributed among all the gas and all the surfaces in accordance with the geometrical configuration of the various surfaces and the absorbing power of the gas, and that portion incident on the surfaces is again partially absorbed and partially reflected, to repeat the process *ad infinitum.*

The technique for making allowance for this complicated problem is an extension of one presented (*5*) for the one-gas-zone, multisurfaced enclosure. As in the earlier published treatment where the direct interchange between two zones was equal to the difference in emissive powers multiplied by $'A_1F_{12}\tau_{12}$ when the zones were surface elements, or a gas-emissivity–area product when one element was the gas zone, so in the present case is the direct interchange equal to a term $\overline{s_1s_2}$, $\overline{g_1g_2}$, $\overline{g_1s_2}$, etc. Similarly, if one allows for the multiple reflections within the enclosure, in the earlier treatment the net interchange per unit difference in emissive powers became $A_1\mathfrak{F}_{12}$ or $A_1\mathfrak{F}_{1G}$; in the present case it may be defined as $\overline{S_1S_2}$, $\overline{G_1S_2}$, or $\overline{G_1G_2}$. And just as it was possible to obtain the values of $A\mathfrak{F}$ from the values of AF, again one can evaluate such terms as $\overline{SS}$, $\overline{GS}$ or $\overline{GG}$ from the terms $\overline{ss}$, $\overline{gs}$, or $\overline{gg}$. Note that the terms $\overline{SS}$, $\overline{GS}$ and $\overline{GG}$, like their lower-case counterparts and like the older $A\mathfrak{F}$, all have the dimensions of area; they will be referred to as *total-interchange area* terms. Note also that, whereas a direct-interchange area specific to two particular zones can be evaluated from knowledge of these two zones alone, the total-interchange area applicable to the same two zones will require knowledge of all zones that they can "see," either directly or by wall reflection.

It is plain that the interchange area between any two zones in the enclosure cannot be a function of temperature as long as the physical properties of the system (emissivities and absorptivities) are independent of temperature. With this limitation, one is justified in evaluating the interchange-area terms at any convenient temperature, and the result will be completely general. The problem of evaluating these terms is simplified if one assigns a value of absolute zero to the temperature of *every surface and gas zone except* ONE, as a consequence of which the emissive powers of every zone except one are all zero. This necessitates an imagined energy withdrawal from all zones, including the gas zones, by means unspecified but not interfering with transmission. Furthermore, one can assign a temperature to the one emitting zone that will make its black emissive power 1; i.e., so that $\mu^2\sigma T^4$ is 1. As a result of maintaining this temperature at the single surface or gas zone, there will be a radiant flux at and toward every surface in the system, and at and away from it as well, owing solely to original emission from the one zone and to the multiple reflections within the enclosure. In a black system this reflected flux would be equal to zero. The terminology to describe this outgoing flux density will be R, with a presubscript designating the original source of the energy and a final subscript the reflecting surface. Thus if gas zone g_1 were the only emitter in the system, the flux densities leaving surfaces s_1, s_2, $\cdots$ would be indicated by ${}_{g_1}R_{s_1}$, ${}_{g_1}R_{s_2}$, $\cdots$. Similarly, if surface s_1 were the only original emitter, the flux density leaving surfaces s_2, s_3 $\cdots$ would be ${}_{s_1}R_{s_2}$, ${}_{s_1}R_{s_3}$, $\cdots$ but the flux density leaving surface s_1 would be equal to ${}_{s_1}R_{s_1}$, owing to reflections, plus ϵ_{s_1} to allow for original emission from the surface.

1. Original Emitter Is a Gas Zone

If, for example, gas zone g_i is the sole original emitter, then the total radiant flux at and away from any surface such as s_j in the system is designated as ${}_{g_i}R_{s_j}\cdot A_{s_j}$. Since this is the reflected flux, multiplication by the ratio of absorptivity (equal to emissivity) of the surface to the reflectivity of the surface (the complement of absorptivity) yields the rate of energy absorption at the surface; and since the value of $\mu^2\sigma T^4$ of the original emitter was made equal to unity, this must by definition be identical with the desired total-interchange area $\overline{G_iS_j}$; or

$$\overline{G_iS_j} = \overline{S_jG_i} = {}_{g_i}R_{s_j}\cdot\frac{A_{s_j}\epsilon_{s_j}}{\rho_{s_j}} \qquad (16)$$

where the reflectivity $1 - \alpha_{s_j}$ has been designated by the single term ρ_{s_j} because it will appear so often. The problem of finding $\overline{G_iS_j}$ has thus been reduced to one of determining the reflected flux ${}_{g_i}R_{s_j}$. To do that one must solve the system of simultaneous equations resulting from writing a radiant energy balance on each surface. The total rate of energy impingement on surface s_1 is equal to the contributions from all the surfaces in the system, including itself, plus the energy it receives directly from the single gas zone which is the original emitter. If the sum of all these terms is then multiplied by the surface reflectivity, the result, which is the reflected flux, must be equal to ${}_{g_i}R_{s_1}\cdot A_{s_1}$. This summation can be carried out for every surface in the system, yielding a set of linear equations, one for each surface, as follows:

$$\begin{array}{l}
(\overline{s_1s_1}\cdot{}_{g_i}R_{s_1} + \overline{s_1s_2}\cdot{}_{g_i}R_{s_2} + \overline{s_1s_3}\cdot{}_{g_i}R_{s_3} + \cdots \overline{g_is_1})\rho_{s_1} = A_{s_1}\cdot{}_{g_i}R_{s_1}\\
(\overline{s_1s_2}\cdot{}_{g_i}R_{s_1} + \overline{s_2s_2}\cdot{}_{g_i}R_{s_2} + \overline{s_2s_3}\cdot{}_{g_i}R_{s_3} + \cdots \overline{g_is_2})\rho_{s_2} = A_{s_2}\cdot{}_{g_i}R_{s_2}\\
(\overline{s_1s_3}\cdot{}_{g_i}R_{s_1} + \overline{s_2s_3}\cdot{}_{g_i}R_{s_2} + \overline{s_3s_3}\cdot{}_{g_i}R_{s_3} + \cdots \overline{g_is_3})\rho_{s_3} = A_{s_3}\cdot{}_{g_i}R_{s_3}\\
\qquad\vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\qquad\qquad\quad \vdots\quad\vdots \qquad\quad \vdots
\end{array} \qquad (17)$$

These may be expressed more compactly as

$$\begin{array}{l}
\sum_j \overline{s_1s_j}\cdot{}_{g_i}R_{s_j} - \dfrac{A_{s_1}\cdot{}_{g_i}R_{s_1}}{\rho_{s_1}} = -\overline{g_is_1}\\
\sum_j \overline{s_2s_j}\cdot{}_{g_i}R_{s_j} - \dfrac{A_{s_2}\cdot{}_{g_i}R_{s_2}}{\rho_{s_2}} = -\overline{g_is_2}
\end{array} \qquad (17a)$$

The solution for any one of the unknowns, ${}_{g_i}R_{s_j}$, may be expressed in determinant form as

$${}_{g_i}R_{s_j} = \frac{{}_{g_i}D_{s_j}}{D} \qquad (18)$$

where the determinant D in the denominator is given by

$$D = \begin{vmatrix}
\overline{s_1s_1} - \dfrac{A_{s_1}}{\rho_{s_1}} & \overline{s_1s_2} & \overline{s_1s_3} \cdots \\
\overline{s_1s_2} & \overline{s_2s_2} - \dfrac{A_{s_2}}{\rho_{s_2}} & \overline{s_2s_3} \cdots \\
\overline{s_1s_3} & \overline{s_2s_3} & \overline{s_3s_3} - \dfrac{A_{s_3}}{\rho_{s_3}} \cdots \\
\vdots & \vdots & \vdots
\end{vmatrix} \qquad (19)$$

and ${}_{g_i}D_{s_j}$ is formed from D by replacing its s_j column (the one having j common to the subscripts of all elements) by the terms $-\overline{g_is_1}$, $-\overline{g_is_2}$, $-\overline{g_is_3}$, $\cdots$. It is to be noted that the value of D is independent of which gas volume is the original emitter, although the value of ${}_{g_i}D_{s_j}$ is not.

Returning now to the expression for the term $\overline{G_iS_j}$ in Equation (16) and substituting the value of ${}_{g_i}R_{s_j}$ from Equation (18), one obtains as the general expression for $\overline{G_iS_j}$ or $\overline{S_jG_i}$,

$$\overline{G_iS_j} \equiv \overline{S_jG_i} = \frac{A_{s_j}\epsilon_{s_j}}{\rho_{s_j}}\cdot\frac{{}_{g_i}D_{s_j}}{D} \qquad (20)$$

From the value of the reflected flux density at the various surfaces it is also possible to calculate the total-interchange area between any two gas cubes. If gas zone g_m is the original emitter, then the total reception at any other gas zone such as zone g_n is equal to $\overline{g_mg_n}$, the direct radiation from g_m to g_n, plus the sum of the products of the reflected flux at each surface in the system multiplied by the fraction of that flux which reaches, and is absorbed by, gas zone g_n. Again, since the black-body emission from zone

g_m has been set at 1, this sum represents the desired $\overline{G_mG_n}$; or

$$\overline{G_mG_n} = \overline{g_mg_n} + \sum_i ({}_{g_m}R_{s_i})(\overline{s_ig_n}) \quad (21)$$

2. Original Emitter Is a Surface Zone

In order to obtain the one remaining kind of total-interchange area, that between two surfaces, it is convenient to assign a temperature to one of the surface zones that will permit its black-body emission to become equal to 1 and to let all other surfaces as well as all gas zones be at absolute zero. One can again write energy balances on each surface in the system, the only changes being that there will be no direct radiation from the gas and that the surface that is the original emitter will send out, in addition to the reflected-flux term, an amount of energy per unit area equal to its emissivity. The set of resultant equations for this case, when surface s_1 is the sole original emitter, is

$$\begin{aligned}
&(\overline{s_1s_1}({}_{s_1}R_{s_1} + \epsilon_{s_1}) + \overline{s_1s_2}\cdot{}_{s_1}R_{s_2} + \overline{s_1s_3} \\
&\quad \cdot{}_{s_1}R_{s_3} + \cdots)\rho_{s_1} = A_{s_1}\cdot{}_{s_1}R_{s_1} \\
&(\overline{s_1s_2}({}_{s_1}R_{s_1} + \epsilon_{s_1}) + \overline{s_2s_2}\cdot{}_{s_1}R_{s_2} + \overline{s_2s_3} \\
&\quad \cdot{}_{s_1}R_{s_3} + \cdots)\rho_{s_2} = A_{s_2}\cdot{}_{s_1}R_{s_2} \qquad (22) \\
&(\overline{s_1s_3}({}_{s_1}R_{s_1} + \epsilon_{s_1}) + \overline{s_2s_3}\cdot{}_{s_1}R_{s_2} + \overline{s_3s_3} \\
&\quad \cdot{}_{s_1}R_{s_3} + \cdots)\rho_{s_3} = A_{s_3}\cdot{}_{s_1}R_{s_3}
\end{aligned}$$

[This will be recognized as identical, except for nomenclature, to the system of equations covering the published case (*5*) of a multiple-surface-zone–single-gas-zone system.] The solution for any one of the unknowns ${}_{s_i}R_{s_j}$ is

$$ {}_{s_i}R_{s_j} = \frac{{}_{s_i}D_{s_j}}{D} \quad (23)$$

where D is the same as the expression obtained earlier and given in Equation (19) and ${}_{s_i}D_{s_j}$ is obtained by substituting into the s_j column of D the terms $-\epsilon_{s_i}\cdot\overline{s_is_1},\ -\epsilon_{s_i}\cdot\overline{s_is_2},\ -\epsilon_{s_i}\cdot\overline{s_is_3}\ \cdots$.

If this value of the reflected-flux density is multiplied by the area of surface s_j and then by the ratio of the absorptivity to the reflectivity, one obtains the absorption rate at surface s_j due to original emission from surface s_i. This is equal to the desired quantity $\overline{S_iS_j}$, since the black emissive power of the original emitter was made equal to 1; or

$$\begin{aligned}
\overline{S_iS_j} &= \frac{A_{s_j}\epsilon_{s_j}}{\rho_{s_j}}\cdot\frac{{}_{s_i}D_{s_j}}{D} \\
&\equiv \frac{A_{s_i}\epsilon_{s_i}}{\rho_{s_i}}\cdot\frac{{}_{s_j}D_{s_i}}{D} = \overline{S_iS_j}\ * \quad (24)
\end{aligned}$$

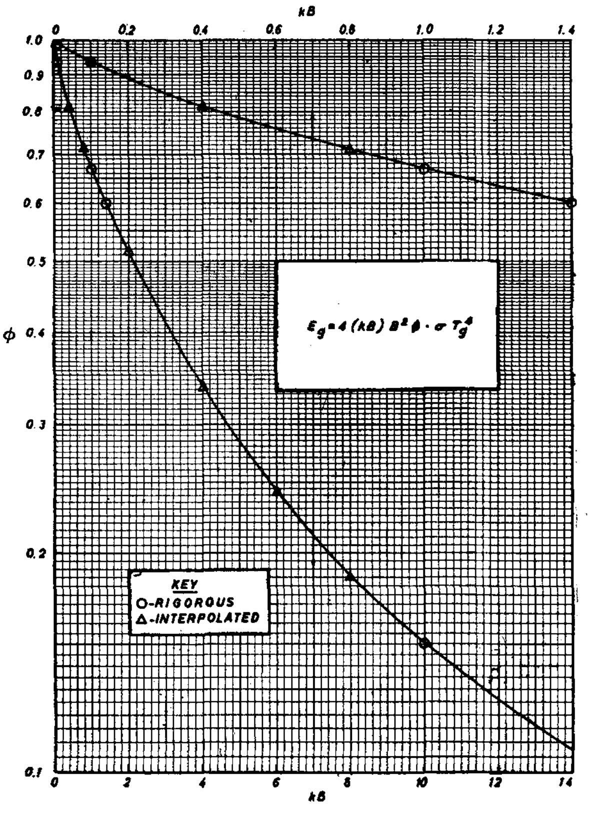

Fig. 1. Escape factor ϕ for emission from a cube of edge B filled with gas of absorption coefficient k.

Two important consequences of the definitions of these three kinds of total-interchange areas, analogous to Equations (13) and (15), are

$$\sum_i \overline{S_jS_i} + \sum_i \overline{S_jG_i} = \epsilon_{s_j}\cdot A_{s_j} \quad (25)$$

$$\sum_i \overline{G_jG_i} + \sum_i \overline{G_jS_i} = 4kV_{g_j} \quad (26)$$

In the most general case, the order of the determinants to be evaluated is equal to the number of surfaces present; the total number of sets of them to be evaluated as a consequence of letting each gas zone and surface zone be original emitters is equal to the total number of zones, gas and surface, in the enclosure; each set involves the evaluation of a number of determinants equal to the total number of surface zones in the system, and, in addition, one must evaluate the common-denominator determinant D once. An alternative procedure for the solution of sets of equations like (17) and (22) would be the use of the Crout Method (*2*) or of high-speed computing machines.

ALLOWANCE FOR A REAL GAS

A real gas, in contrast to the hypothetical uniformly absorbing gray gas used in the preceding derivations, exhibits a variation in absorption coefficient with wave length, which affects the validity of the method presented in several ways.

1. The total emissivity or absorptivity of the gas does not conform to the Beer's-Law exponential expression which characterizes monochromatic or gray-gas radiation and therefore does not approach 1 as a limit as the path length is increased.
2. Since by Planck's Law the wave-length range of interest depends on temperature, total gas emissivity is temperature dependent.
3. Gas absorptivity also depends on temperature.
4. Space variations in composition of the gas may cause local variations in its absorptivity and emissivity.

It has been shown (*5*) that, since the variation, with path length x, of the emissivity or absorptivity of a real gas is representable by the weighted sum of a number of e functions

$$\sum_i a_i(1 - e^{-k_ix}), \quad \text{with} \quad \sum a_i = 1,$$

it follows that the contribution of the real gas to radiant flux in an enclosure is the weighted sum of the independent contributions of several gray gases of different absorption coefficients k_i, weighted in proportion to the coefficients a_i. It has also been shown (*5*) that for one-gas-zone systems the calculated values of radiant flux are generally of adequate numerical accuracy when based on the assumption that the real gas is replaceable by but two "gray" gases one of which has an absorption coefficient k

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*Easier to use but the equivalent of this formulation is the statement that

$$\overline{S_iS_j} = \frac{A_{s_j}\epsilon_{s_j}}{\rho_{s_j}}\cdot\frac{A_{s_i}\epsilon_{s_i}}{\rho_{s_i}}\cdot\frac{D'_{s_is_j}}{D}$$

where $D'_{s_is_j}$ is the cofactor of row s_i and column s_j, defined as $(-1)^{i+j}$ times the minor of D formed by crossing out the s_i row and s_j column of D. This is correct except when $i = j$, for which case

$$\overline{S_iS_i} = \frac{A_{s_i}\epsilon_{s_i}}{\rho_{s_i}}\left(\epsilon_{s_i} + \frac{A_{s_i}\epsilon_{s_i}}{\rho_{s_i}}\right)\frac{D'_{s_is_i}}{D}.$$

and a weighting factor a and the other an absorption coefficient of 0 (i.e., a "clear" gas) and a weighting factor of $1 - a$. (These values of k and a, however, are not to be considered intrinsic properties of the gas; they must be fitted to the problem at hand.) For the case of a gas composed of a single gray and a clear component the total emissivity and absorptivity are related to the path length x by

$$\epsilon_x = a(1 - e^{-kx}) \qquad (27)$$

and the unattenuated emission per unit volume is changed from $4k(\mu^2\sigma T^4)$ to $4ka(\mu^2\sigma T^4)$. If one fits the exponential above to the actual emissivity–vs.–path-length curve for a real gas at three path lengths—namely, zero path length, a path length equal to the primary path length L for the system (for which emissivity equals ϵ_L), and a path length twice this value (for which emissivity equals ϵ_{2L})—it may be shown (*5*) that a and kL are given by the relations

$$a = \frac{(\epsilon_L)^2}{2\epsilon_L - \epsilon_{2L}} \qquad (28)$$

and

$$kL = \ln \frac{\epsilon_L}{\epsilon_{2L} - \epsilon_L}$$
$$= -\ln (1 - \epsilon_L/a) \qquad (29)$$

If because of high diathermancy of the gas and/or high wall reflectivity it is desirable to fit the ϵ-x curve at 0 and L but at nL instead of $2L$, it can readily be shown that instead of (28), a is obtained from the relation

$$1 - \epsilon_{nL}/a = (1 - \epsilon_L/a)^n \qquad (30)$$

and kL is obtained as before from relation (29). For the case of $n = 3$, (30) yields

$$a = \frac{2\epsilon_L}{3 - \sqrt{4(\epsilon_{3L}/\epsilon_L) - 3}} \qquad (31)$$

As a consequence of this method of handling a real gas the total-interchange areas to be used become

$$\overline{SS} = a[\overline{SS}]_k + (1 - a)[\overline{SS}]_{k=0} \qquad (32)$$

$$\overline{GS} = a[\overline{GS}]_k \qquad (33)$$

$$\overline{GG} = a[\overline{GG}]_k \qquad (34)$$

Total gas emissivity is temperature dependent, primarily because of the Planck's Law shift with temperature in the wave lengths of importance, secondarily because the absorption coefficients of the gas are themselves dependent on temperature. The numerical effect of this variation can be expressed, in conformity with the treatment of section 1 just preceding, as an effect on a and k. For the particular case of gas radiation due only to water vapor and carbon dioxide at equal partial pressures, it is found that over the temperature range of 1,000° to 3,500°R. (1) k is constant to within ±4% at values of PL below 0.4 foot-atmosphere and to within ±30% at a value of 1 ft.-atmosphere and (2) a decreases by about a factor of 2 at all values of $\sum PL$ up to 1 ft.-atmosphere when the temperature is increased 3.5-fold, or decreases 20 to 30% for a temperature change of 1,000°F. The variation in kL is tolerable, particularly at low values of $\sum PL$; the variation in a is less acceptable and stands as a source of error in use of the method on a furnace with large space variation in gas temperature. Greatest weight should be given to the a of that part of the system responsible for the major heat transfer. It should be emphasized here that choice of a and k to conform to conditions in that part of the enclosure which contributes most of the heat transfer will yield a solution the range of error of which is but a small fraction of the range of error in a and k mentioned above.

In principle, it is possible to go one step further toward a rigorous solution. When the total-interchange-area terms from a given zone, say g_1, to each of the others in the system are evaluated with a k based on the temperature of that zone, the error introduced cannot be large, as indicated by the following. The interchange with adjacent zones will be nearly correct since their temperature and hence their k will not be very far from the values of zone g_1; the interchange area with a zone very far away and consequently at a considerably different temperature will be in error, but because the term itself is so small one can tolerate a very substantial error in it. It would be possible to go through the procedure of evaluating total-interchange areas, letting each zone in turn be the original emitter in the system, and evaluating each set of areas at the k of the emitting zone. One result of such a technique would be that $\overline{G_1G_2}$ would be different from $\overline{G_2G_1}$ since the two are based on k's evaluated at T_{g_1} and T_{g_2}, respectively. To carry out such a procedure it would be necessary first to guess all the temperatures in the system and then, after a first solution to the problem, to revise the estimates of temperature and begin all over if the original assumptions were very much in error. Furthermore, the amount of work required to calculate the total-interchange areas would be increased considerably since the determinant D in Equations (18), (20), (23), and (24) would have to be completely reevaluated as each zone in turn became the original emitter. Although in principle one could more closely approach a rigorous solution by this technique, the labor involved would be prohibitive; and in engineering use one would generally introduce the approximation that gas emission and absorption properties can be evaluated at a single mean temperature.

When the mean surface temperature is either very high or very low compared with the gas temperature, one of the two processes, gas emission to the surface or gas absorption of surface emission, dominates the interchange. The use of an a and k corresponding to the dominant mechanism then yields numerically satisfactory results. If, for example, the surface is relatively cold the method will allow incorrectly for absorption, by the gas, of original surface emission, but the effect on the net flux is unimportant; and correct allowance will be made for self-absorption in the gas due to wall reflections. The surface satisfies this condition when its emissive power is that of a black surface at less than two-thirds the absolute gas temperature.

Gas absorptivity depends primarily on the temperature of the emitter, secondarily on that of the gas, because absorption coefficients depend somewhat on temperature. Except where gas and surface temperatures are the same, gas absorptivity of surface radiation and gas emissivity are different. By the very nature of the calculational technique being developed, however, one is forced into using for the entire system a single pair of mean values for a and k which will be able to account for both emission and absorption at all the temperature levels of importance in the enclosure. The error so introduced can be minimized by some such procedure as the following.

Although interchange between a uniform-temperature gas mass and surrounding black walls at a uniform temperature actually involves the difference between gas emission and gas absorption, it may be satisfactorily approximated by a simpler expression when the surface temperature is no less than two-thirds the gas temperature. That expression (*5*) is

$$q_{g\rightleftarrows s} = \epsilon_{g,T_{avg}} \cdot K \cdot \sigma(T_g^4 - T_s^4) \qquad (35)$$

where $\epsilon_{g,T_{avg}}$ is the gas emissivity evaluated at the arithmetic mean of the gas and surface temperatures and where

$$K = \frac{4 + d + b - c}{4} \qquad (36)$$

For evaluating K, d and b are the fractional changes in gas emissivity per unit fractional change in PL and T_g, respectively (i.e., $\partial \ln \epsilon_g/\partial \ln PL$ and $\partial \ln \epsilon_g/\partial \ln T_g$), and c is an empirical constant (*4, 5*) used in evaluating absorptivity from emissivity, equal to 0.65 for carbon dioxide and 0.45 for water vapor.

For a gray rather than black-walled system it may be shown (Appendix 4) that the term $\epsilon_{g,T_{avg}} K$ of Equation (35) is replaced by the infinite series

$$\epsilon_s^2 \sum_{n=1}^{\infty} \rho_s^{n-1} K \epsilon_{g,nL,T_{avg}}$$

This is identical with the result obtained when the method of this paper is applied to the single-gas-zone single-wall-zone case, provided a single pair of values of a and k is used corresponding to an effective gas emissivity $K\ \epsilon_{g,T_{avg}}$ rather than ϵ_g. Accordingly, a single value for emissivity and absorptivity, equal to $K\ \epsilon_{g,T_{avg}}$ [where $T_{avg} = (T_s + T_g)/2$] and evaluated at the two path lengths necessary to determine a and k, may be used to represent satisfactorily both emission and absorption. Choice of representative mean temperatures T_g and T_s to use in these relations remains a problem. For gases containing carbon dioxide and water in equal proportions, a 200°F. error in choice of T_{avg} produces a 10 to 20% error in $K\epsilon_g$. It is recommended that for any furnace enclosure in which the gas temperature varies enough to make a choice of representative temperature doubtful, a preliminary solution of the problem be carried out by means of the one-gas-zone technique already available (*5*) and that the resulting temperature be used for T_g.

Limited allowance for the effect of local variations in concentration of the absorbing or emitting components of the gas may be made. Appendix 5 presents an approximation for that case in which the fluctuation in concentration produces local changes only in k and not in a.

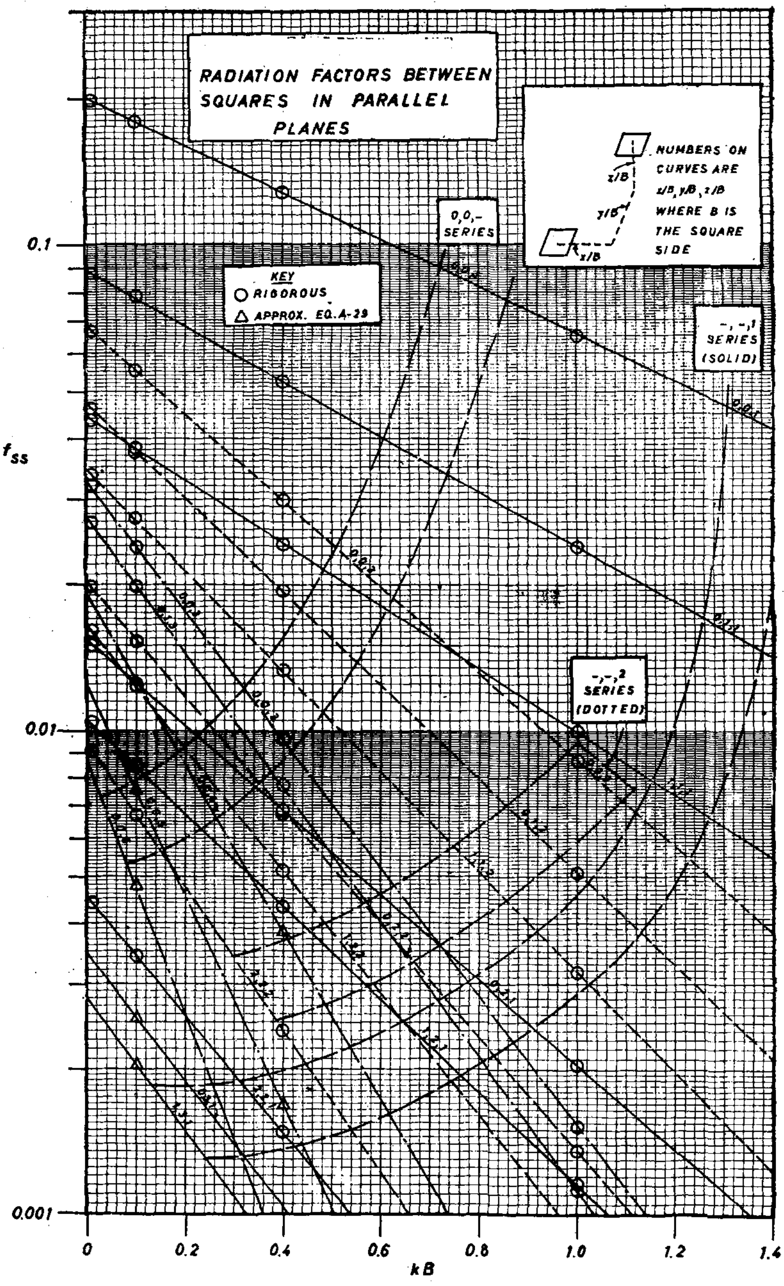

Fig. 2. Reception factor f_{ss} between squares of side B in perpendicular planes (f_{ss} = direct-view factor × transmittance).

ENTHALPY FLUX, CONVECTIVE FLUX, AND STORAGE

The only remaining terms to be evaluated for use in the total-energy balance on each zone are the nonradiation terms, enthalpy flux, convective flux and, when conditions are not steady in time, energy storage. If detailed knowledge of the gas composition, the direction of flow, and the molal flow rate G_m per unit area is available for every point in the enclosure, the net energy transferred across a boundary between any two gas zones may be evaluated. It is

$$\int_A G_{m_n}(H_s + H_c)\, dA \qquad (37)$$

where G_{m_n} is the component of G_m normal to dA, and H_s and H_c are the molal sensible enthalpy and the molal chemical enthalpy of the stream, both evaluated at the temperature of the zone from which the gas flows.

Similarly, the *one-way* convective flux to or from a gas or surface zone may be written as

$$\int_A h_c T\, dA \qquad (38)$$

where the temperature to be used (gas or surface) depends on the source of the energy and h_c is the convection coefficient. For transport between gas and surface, values of h_c are given by McAdams (*6*); for transport between adjacent gas zones (when significant compared with transport by the mean or bulk flow) an estimate of the coefficient may be made from the eddy diffusivities in pipes and jets (*3, 7*).

The energy storage rate in a zone is its heat capacity C_s or C_g times $\dot{T}$, the rate of temperature rise.

Since all the radiation terms in the energy balances are given as multiples of T^4, it is desirable to linearize the equations in T^4 by forcing the convection and enthalpy-flux terms into a fourth-power law. A new coefficient may be defined such that when multiplied by σT^4 (hereafter termed W for simplicity) it will yield the one-way flux between any two zones. If the enthalpy flux and convective flux coefficients are denoted by η and γ respectively, with a first subscript representing the energy-source zone and a second one the energy-sink zone, they become

$$\eta_{g_1g_i} = \frac{\int_A G_{m_n}(H_s + H_c)_{T_{g_i}}\, dA}{W_{g_1}} \qquad (39)$$

$$\gamma_{g_1g_i} = \frac{\int_A h_c T_{g_1}\, dA}{W_{g_1}} \qquad (40)$$

and

$$\gamma_{s_1g_i} = \frac{\int_A h_c T_{s_1}\, dA}{W_{s_1}} \qquad (41)$$

where A is the area common to zones 1 and i and G_{m_n} is the normal flow, from zone 1 to zone i, through that area.

Terms $\eta_{g_1 s_i}$ will appear only when a surface s_i sharing an area with g_1 contains ports through which gas flows, causing enthalpy flux out of the system. The coefficients η and γ have the dimensions of area.

FORMULATION OF ENERGY BALANCES

With the convention that a positive q represents flux to an element in question, the energy interchange between gas zone g_1 and all other zones (g_2, g_3, $\cdots$ g_i, s_1, s_2, $\cdots$ s_j, those having a common boundary with g_1; g_{10}, g_{11}, $\cdots$, s_{10}, s_{11}, $\cdots$ those not touching g_1) will take the form

$$\begin{aligned} & W_{g_1}(\overline{G_1G_1} - 4kaV_{g_1} - \sum_i \gamma_{g_1g_i} \\ & - \sum_j \gamma_{g_1s_j} - \sum_i \eta_{g_1g_i} - \sum_j \eta_{g_1s_j}) \\ & + W_{g_2}(\overline{G_2G_1} + \gamma_{g_2g_1} + \eta_{g_2g_1}) \\ & + W_{g_3}(\overline{G_3G_1} + \gamma_{g_3g_1} + \eta_{g_3g_1}) + \cdots \\ & + W_{g_{10}}(\overline{G_{10}G_1}) + W_{g_{11}}(\overline{G_{11}G_1}) + \cdots \\ & + W_{s_1}(\overline{S_1G_1} + \gamma_{s_1g_1}) \\ & + W_{s_2}(\overline{S_2G_1} + \gamma_{s_2g_1}) + \cdots \\ & + W_{s_{10}}(S_{10}G_1) + W_{s_{11}}(S_{11}G_1) + \cdots \\ & + \dot{M}(H_c + H_s)_{in} - C_{g_1}\dot{T}_{g_1} = 0 \quad (42) \end{aligned}$$

In this expression $\sum_i \gamma_{g_1g_i}$ and $\sum_j \gamma_{g_1s_j}$ represent the one-way convection from zone g_1 to every zone with which it shares an area; $\sum_i \eta_{g_1g_i}$, however, represents the sum of enthalpy fluxes only for those zones g_i into which gas flows *from* g_1. Where gas flow is *into* g_1 from g_i, the enthalpy flux term, which will be $\eta_{g_ig_1}$, will appear as part of the coefficient of W_{g_i}. For simplification of notation for use in a later step, the following substitute notation will be used:

$$\sum_i \gamma_{g_1g_i} + \sum_j \gamma_{g_1s_j} = \gamma_{g_1} \quad (43)$$

and

$$\sum_i \eta_{g_1g_i} + \sum_j \eta_{g_1s_j} = \eta_{g_1} \quad (44)$$

The next-to-the last term of Equation (42) is present only where one of the bounding surfaces of zone g_1 contains ports through which there enters, at a molal flow rate $\dot{M}$ from outside the system, some gas and fuel having a molal sensible-plus-chemical enthalpy, $(H_c + H_s)_{in}$. The last term is absent in steady state problems and insignificant compared with the others in any case where g represents gas. However, g might represent the glass in a problem of glass heating, and the energy-storage term is then significant.

Similarly, a balance on any surface zone s_1 representing the interchange between it and all other zones (g_1, g_2, $\cdots$, those having a common boundary with s_1; g_{10}, g_{11} , $\cdots$, those not touching s_1; and s_2, s_3, $\cdots$) takes the form

$$\begin{aligned} & W_{g_1}(\overline{G_1S_1} + \gamma_{g_1s_1}) \\ & + W_{g_2}(\overline{G_2S_1} + \gamma_{g_2s_1}) + \cdots \\ & + W_{g_{10}}(\overline{G_{10}S_1}) + W_{g_{11}}(\overline{G_{11}S_1}) + \cdots \\ & + W_{s_1}(\overline{S_1S_1} - \epsilon_{s_1}A_{s_1} - \sum_i \gamma_{s_1g_i}) \\ & + W_{s_2}(\overline{S_2S_1}) + W_{s_3}(\overline{S_3S_1}) + \cdots \\ & - q_{s_1} - C_{s_1}\dot{T}_{s_1} = 0 \qquad (45) \end{aligned}$$

where the next-to-last term q_{s_1} represents the net outward heat flux through the wall in question. It is to be remembered that for a true heat sink of known temperature this wall flux will be an unknown quantity; for a well-insulated refractory surface the wall flux will be 0 but the temperature will be unknown. For a billet being heated in batch operation q_{s_1} will be 0 but $C_{s_1}\dot{T}$ will not; if the operation is continuous and in dynamic equilibrium, either of the terms may be used to describe the transfer. (See later.) Again, to simplify nomenclature the term $\sum_i \gamma_{s_1g_i}$ will be represented by γ_{s_1}.

To aid in handling the solution of the system of energy balances of which Equations (42) and (45) are representative, the coefficients on W's may be arranged in a matrix, appearing in Table 1. Each row contains the coefficients of W's in a single energy balance, the rows from top to bottom corresponding to balances on g_1, g_2 $\cdots$, s_1, s_2 $\cdots$ in sequence. The coefficients are arranged

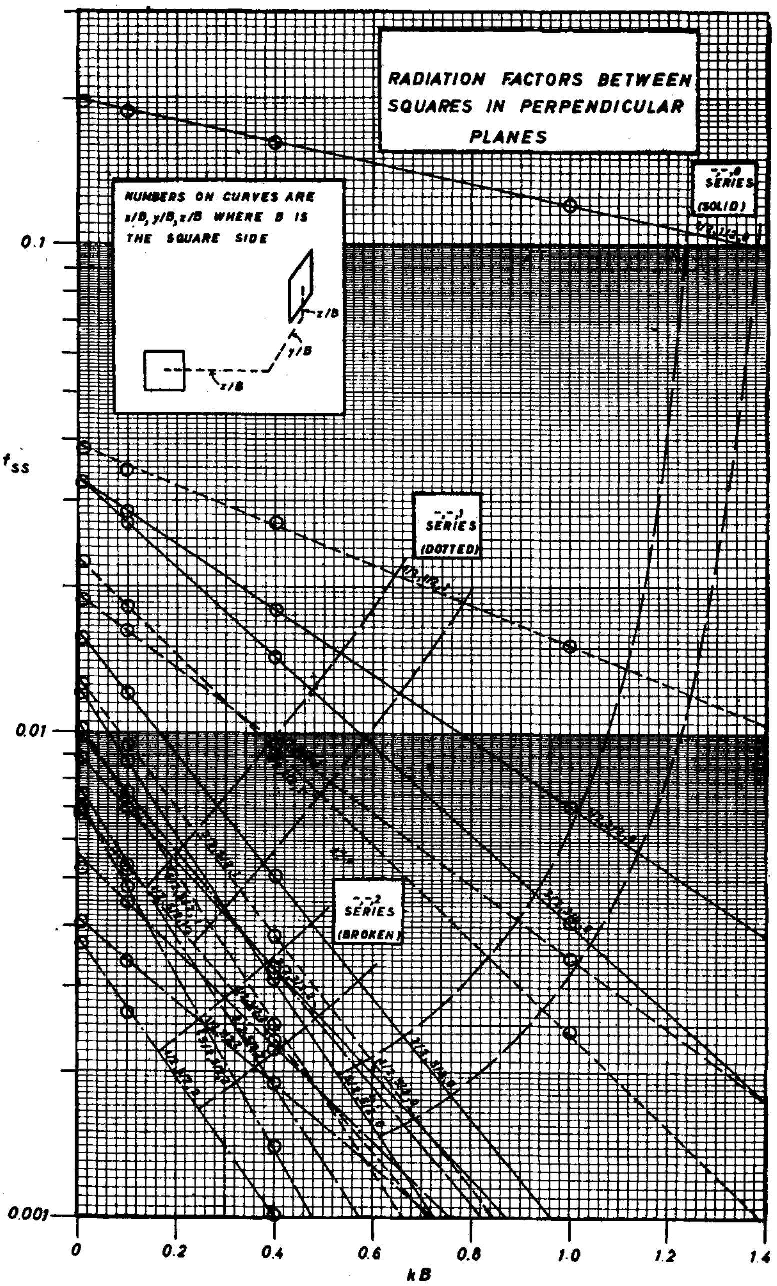

Fig. 3. Reception factor f_{ss} between squares of side B in perpendicular planes (f_{ss} = direct-view factor × transmittance).

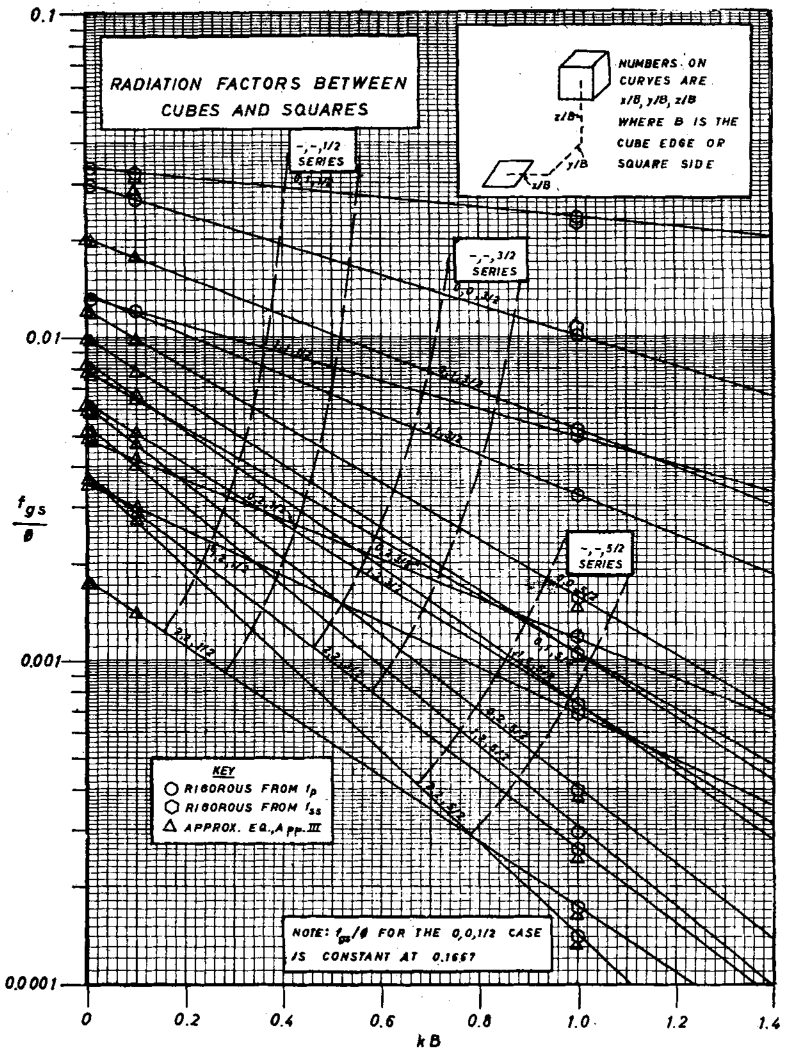

Fig. 4. Reception factor f_{gs} from cube to square, both of sides B (f_{gs} = direct-view factor × transmittance; ϕ from Figure 1).

in columnar form to make coefficients on $W_{g_1}, W_{g_2} \cdots W_{s_1}, W_{s_2}$ appear in sequence, left to right. Recapitulating, the energy balance on any gas zone is formed by summing the products formed by multiplying the terms in a row by their column headings and equating the result to the quantity $-\dot{M}(H_c + H_s)_{in} + C_g\dot{T}_g$ (either or both of which may be 0); the energy balance on any surface zone is formed by similar product summation for the row in question and equating of the result to $q_s + C_s\dot{T}$.

Before the solution of these equations is considered, it is necessary to examine the possible meanings, in different furnace chambers, of such terms as q_{s_i} appearing in the balance on zone s_i. There are three cases:

1. The temperature T_{s_i} is known and q_{s_i} unknown, as in the case of a surface zone representing a screen of water- or oil-cooled tubes. For this case W_{s_i} is known. If there are g gas zones and s surface zones of which u are in this category, there are still $g + s$ equations and as many unknowns, but there are only $g + s - u$ unknown temperatures and u unknown q's; and one needs to solve simultaneously only $g + s - u$ equations, followed by substitution in the others, to find the q's.

2. Both the temperature T_{s_i} and the flux q_{s_i} are unknown—the case of heat loss through a thin or poorly insulated refractory wall. A solution requires sufficient knowledge of the wall to evaluate q_{s_i} in the form $A_{s_i}U_{s_i}(T_{s_i} - T_0)$, where U_{s_i} is the over-all coefficient of heat transfer, inner wall surface to outside air at T_0. For simplicity in using this relation as a part of the system of fourth-power equations, one writes

$$q_{s_i} = A_{s_i}U_{s_i}(T_{s_i} - T_0) = \left(\frac{A_{s_i}U_{s_i}}{\sigma T_{s_i}^{\,3}}\right)W_{s_i} - A_{s_i}U_{s_i}T_0 \qquad (46)$$

Since q_{s_i} was the sole term on the right side of the energy balance formed from row s_i of the matrix (Table 1), the first term on the right side of (46) is transposed and the coefficient in parentheses incorporated, with sign changed, into the term in the W_{s_i} column of row s_i, leaving $-A_{s_i}U_{s_i}T_0$ as a known quantity on the right.

3. The temperature T_{s_i} and flux q_{s_i} are unknown, but related by an enthalpy balance—the case of a continuous furnace with stock, forming zone s_i, entering with hourly heat capacity $\dot{w}C_p$ from zone s_j or from outside the system with a specified temperature. For this case

$$q_{s_i} = \dot{w}C_p(T_{s_i} - T_{s_j}) \equiv \left[\frac{\dot{w}C_p}{\sigma T_{s_i}^{\,3}}\right]W_{s_i} - \left[\frac{\dot{w}C_p}{\sigma T_{s_j}^{\,3}}\right]W_{s_j} \qquad (47)$$

As before, the coefficients in brackets are to be incorporated, with their signs changed, into the W_{s_i} and W_{s_j} columns of s_i; and the right-hand side of the energy balance is 0. Terms of the energy balance on s_j must be similarly modified.

4. Hybrids can exist, for example billets on water-cooled skids, involving the mechanisms of both classes (b) + (c); their treatment along the lines above is obvious.

Unless computers are available for handling higher order determinants, it is not recommended that a determinant technique be used to solve for the unknown temperatures for two reasons:

1. The various coefficients have profoundly different magnitudes because of the poor "view" that some zones have of other remote ones, in consequence of which one or two terms in any one of the energy balances may be of overriding importance.
2. Some of the terms in the "known" coefficients of Table 1 are in fact unknown because they have been obtained by a forced linearization and must therefore be estimated and changed when found not to conform to the temperatures obtained from the solution.

On the basis of experience in handling numerical solutions the best procedure is to guess all the temperatures (or W's), to insert all but the most important one of the guesses into the first equation and solve for one unknown W, to use this answer and the remaining guesses except for one in the next equation, etc. For those cases tried, the solution converged quite rapidly.

CONCLUSION

A method for predicting radiant heat interchange in enclosures where allowance is to be made for gas-temperature variation has been presented. This includes a set of reception factors permitting evaluation of any exchange problem the geometry of which is represented in cartesian coordinates.

The new technique is capable of solving far more complex problems than could previously have been handled. Since the radiation processes occurring within an enclosure are extremely complicated, the technique presented here which tries to take into account some of those complexities has turned out to be complicated itself. Plainly, however, there are problems the numerical solution of which is of sufficient importance to justify the considerable time required. One important engineering application of the new method is its use on a particular model or class of furnaces in comparison with various engineering shortcuts, to determine the range over which one of them may be relied upon.

APPENDIXES

1. EVALUATION OF ϕ

The rate of emission from an isothermal gas cube may be built up from the simpler case of the radiant interchange between a small-volume element and a plane area. Consider a differential volume of gas located at a distance X measured along the normal through one corner of a rectangle as in Figure 7. The total emission from the differential volume over the total solid angle of 4π is equal to $4k \cdot dV \cdot W$, and the energy received by a differential area on the plane is this times the fractional solid angle, reduced to allow for absorption by the intervening gray gas.

Radiation to differential plane area

$$= 4k \cdot dV \cdot W \cdot e^{-kr} \cdot \frac{d\omega}{4\pi} \qquad \text{(A1)}$$

Division by the emission from the volume and integration over the total solid angle subtended by the rectangle at dV gives the fraction of energy emitted from the differential volume that reaches the plane; this will be termed f_p. Express $d\omega$ in cylindrical coordinates as $\sin \psi \, d\psi \, \Delta\theta$ or $-\Delta\theta \, d(\cos \psi)$. Then

$$f_p = \frac{1}{4\pi} \int_{\cos\psi = 1/\sqrt{1+(\alpha/X)^2+(\beta/X)^2}}^{\cos\psi = 1} \Delta\theta \cdot e^{-kX/\cos\psi} \, d(\cos\psi), \qquad \text{(A2)}$$

where $\Delta\theta$ is the difference of the limiting values of θ associated with the curved-surface element lying at a constant angular position off the axis of X. Numerical values of f_p for a wide range of values of α/X, β/X, and kX have been obtained by graphical integration (*1*).

The factor ϕ, defined as the fraction of the energy originating within a cube of edge B which escapes through the boundaries, is therefore the integral of the sum of a maximum of 24 f_p terms (fraction of energy from infinitesimal volume source reaching walls) integrated over the volume and then divided by the volume, or

$$\phi = \frac{\int_v f_p \cdot dx \, dy \, dz}{B^3} = \int_0^1 \int_0^1 \int_0^1 f_p \cdot d\frac{x}{B} \, d\frac{y}{B} \, d\frac{z}{B} \qquad \text{(A3)}$$

Figure 1 was obtained by triple graphical integrations of Equation (A3).

2. EVALUATION OF RECEPTION FACTORS

The derivation of the equations and details of the graphical integrations for evaluation of the reception factors between any two zones in the enclosure will not be given here, but are available in reference *1*.

Use of nomenclature as shown in Figure 8 and of the same type of reasoning as in the derivation in Appendix 1 yields the factor between two squares of surface, each of side B, lying in either perpendicular or parallel planes:

$$f_{ss}(\text{parallel}) = \frac{1}{\pi} \int_{x_1} \int_{y_1} \int_{x_2} \int_{y_2} \cdot e^{-kB(r/B)} \cdot \frac{(\Delta z/B)^2}{(r/B)^4} \cdot d\left(\frac{x_1}{B}\right) d\left(\frac{y_1}{B}\right) d\left(\frac{x_2}{B}\right) d\left(\frac{y_2}{B}\right) \qquad \text{(A4)}$$

$$f_{ss}(\text{perpendicular}) = \frac{1}{\pi} \int_{x_1} \int_{z_1} \int_{y_2} \int_{z_2} \cdot e^{-kB(r/B)} \cdot \frac{(\Delta x/B)(\Delta y/B)}{(r/B)^4} \cdot d\left(\frac{x_1}{B}\right) d\left(\frac{z_1}{B}\right) d\left(\frac{y_2}{B}\right) d\left(\frac{z_2}{B}\right) \qquad \text{(A5)}$$

where

$$r/B = \left[\left(\frac{x_2 - x_1}{B}\right)^2 + \left(\frac{y_2 - y_1}{B}\right)^2 + \left(\frac{z_2 - z_1}{B}\right)^2\right]^{1/2} \qquad \text{(A6)}$$

Similarly, for interchange between two gas cubes, with nomenclature as given in Figure 9 and with r/B from equation (A6), the factor is given by

$$f_{gg} = \frac{kB}{4\pi} \int_{x_1} \int_{y_1} \int_{z_1} \int_{x_2} \int_{y_2} \int_{z_2} \frac{e^{-kB(r/B)}}{(r/B)^2} \cdot d\left(\frac{x_1}{B}\right) d\left(\frac{y_1}{B}\right) d\left(\frac{z_1}{B}\right) d\left(\frac{x_2}{B}\right) d\left(\frac{y_2}{B}\right) d\left(\frac{z_2}{B}\right) \qquad \text{(A7)}$$

The remaining type of interchange—gas to surface—may be obtained from the f_p plots described in Appendix 1. The fraction of the radiation originating in a gas cube which reaches a surface square is

$$f_{gs} = \frac{\int_v \sum f_p \cdot dx \, dy \, dz}{B^3} \qquad \text{(A8)}$$

$$= \int_0^1 \int_0^1 \int_0^1 \sum f_p \cdot d\left(\frac{x}{B}\right) d\left(\frac{y}{B}\right) d\left(\frac{z}{B}\right)$$

where $\sum f_p$ represents the algebraic sum of four f_p factors yielding the fraction of the radiation from a differential volume element reaching the surface zone.

Multiple graphical integrations of Equations (A4), (A5), (A7), and (A8) yielded the working plots, Figures 2 through 5.

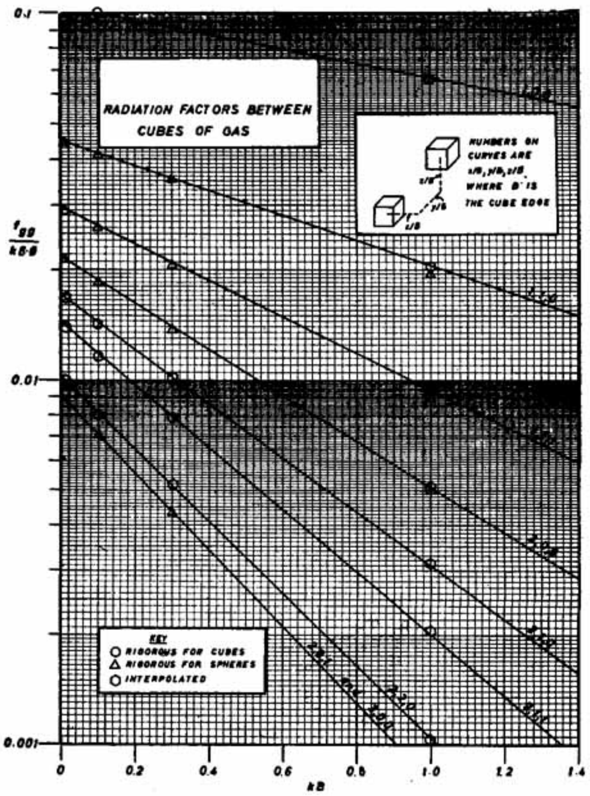

Fig. 5. Reception factor f_{gg} between cubes of side B (f_{gg} = direct-view factor × transmittance × absorptivity; ϕ from Figure 1).

3. LIMITING VALUES OF RECEPTION FACTORS AS r/B INCREASES

As the separating distance between the centers of any two elements becomes very large, so that the view and the path length for absorption are the same for all points in either zone, then the graphical integrations can be dispensed with. The reception factor may be evaluated as the product of the gas transmittance, equal to $e^{-kB(r_c/B)}$ where r_c/B is the center-to-center distance measured in terms of B, and a factor visualizable as the product of a direct-view factor and a fractional absorption in the receiver if a gas. Table 2 summarizes these limiting values.

4. INTERCHANGE BETWEEN GAS AND GRAY WALLS

Consider a gas uniform in temperature and surrounded by gray walls of uniform temperature. Surface absorption of gas emission and gas-absorption of surface emission will each be the sum of an infinite series. It has been shown (*4*) that the net flux may be written

$$q/A]_{g\rightleftarrows s} = \epsilon_s^{\,2} W_g \sum_{n=1}^{\infty} \rho_s^{\,n-1} \epsilon_{g,nL}$$

$$- \epsilon_s^{\,2} W_s \sum_{n=1}^{\infty} \rho_s^{\,n-1} \alpha_{gs,nL} \qquad \text{(A9)}$$

In general, the series summation is made numerically, by use of gas-radiation charts. The members of these two series may be paired to replace (A9) by

$$q/A]_{g\rightleftarrows s} \qquad \text{(A10)}$$

$$= \epsilon_s^{\,2} \sum_{n=1}^{\infty} \rho_s^{\,n-1} (\epsilon_{g,nL} W_g - \alpha_{gs,nL} W_s)$$

If the gas emissivity for path length nL is assumed expressible in the form $a(1 - e^{-knL})$ and absorptivity is obtainable from emissivity by the relation (*5*) $\alpha_{gs,nL} = \epsilon_{g,nLT_s/T_g,T_s}(T_g/T_s)^{\epsilon}$ and if emissivity is proportional, over the temperature range T_g to T_s, to the b power of temperature, it has been shown (*1*) that the terms in parentheses may be written

$$K\epsilon_{g,nL,T_{avg}}(W_g - W_s)$$

where

$$K = \frac{4+b-c-\left(\dfrac{a}{\epsilon_{g,L}}-1\right)\ln\left(1-\dfrac{\epsilon_{g,L}}{a}\right)}{4} \qquad \text{(A11)}$$

It is to be noted that if emissivity is assumed given by $a(1 - e^{-kL})$, then from the definition of d, K in (A11) is the same as K in Equation (36).

The series in (A10) may now be summed

$$q/A]_{g\rightleftarrows s}$$

$$= \epsilon_s^{\,2}(W_g - W_s) \sum_{n=1}^{\infty} \rho_s^{\,n-1}(K\epsilon_{g,nL,T_{avg}})$$

If the effective emissivity $K\epsilon_{g,nL}$ is expressible in the form $a(1 - e^{-knL})$, then the foregoing becomes

$$q/A]_{g\rightleftarrows s}$$

$$= \epsilon_s^{\,2}(W_g - W_s) \sum_{n=1}^{\infty} \rho_s^{\,n-1} a(1-e^{-knL})$$

$$= \epsilon_s^{\,2}(W_g - W_s) a\left(\frac{1}{1-\rho_s} - \frac{e^{-kL}}{1-\rho_s e^{-kL}}\right)$$

and since for gray surfaces $1 - \rho_s = \epsilon_s$, $q/A]_{g\rightleftarrows s}$

$$= (W_g - W_s) a \frac{1}{\dfrac{1}{(\epsilon_{g,T_a}/a)} + \dfrac{1}{\epsilon_s} - 1} \qquad \text{(A12)}$$

This is the very relation obtainable by use of the method of this paper [or the older method (*5*) which is identical to it for the one-gas-zone case], provided that ϵ_g is replaced by $K\epsilon_g$.

5. LOCAL VARIATIONS IN CONCENTRATION OF ABSORBENT

When the concentrations of the absorbing components of the gas vary locally without causing the absorption characteristics or the proportions of the various components to change, allowance can be made for these variations. Knowledge must be available however, concerning the relation between concentration and k. The variation of $\epsilon_{g,L}$ with L, generally expressed graphically, must be available for a known uniform concentration of absorber; and the values of a and k obtained from it, by substituting into (28) and (29) the values of ϵ_g at L and $2L$, must be the same as those obtained at another concentration of absorber, after due allowance for the concentration ratio. In other words the values of a and k_{std}, where k_{std} is k adjusted to a standard concentration of absorber, must not vary. If those conditions are met or a reasonable compromise equivalent to meeting them is accepted, the following argument is applicable.

The transmittance of radiation through a path r with constant k is e^{-kr}; when k varies with r, it is $e^{-k_{avg}r}$, where k_{avg} is given by

$$k_{avg} = \frac{\int_0^r k\,dr}{r} \qquad \text{(A13)}$$

When gas cube 1 is radiating to gas cube 2 and their center-to-center distance of separation r is large compared with the side B of each, their interchange area is given by

$$\overline{g_1 g_2} = 4k_1 B^3 e^{-k_{avg}r} k_2 B^3/\pi r^2 \qquad \text{(A14)}$$

whereas the corresponding relation based on a fixed $k(= k_{avg})$ is

$$\overline{g_1 g_2}]_{k_{avg}} = 4k_{avg}^{\,2} B^6 e^{-k_{avg}r}/\pi r^2 \qquad \text{(A15)}$$

Although the assumptions that emission from g_1 is associated with a fixed k_1 and absorption at g_2 is associated with a fixed k_2 are no longer valid as the center-to-center separating distance approaches B, the use of the ratio of the right-hand sides of (A14) to (A15) to correct the value of $g_1g_2]_{k_{avg}}$, as read from Figure 5, should not lead to much error. Then

$$\overline{g_1 g_2}_{\text{variable } k} = \overline{g_1 g_2}]_{k_{avg}}\left(\frac{k_1 k_2}{k_{avg}^{\,2}}\right) \qquad \text{(A16)}$$

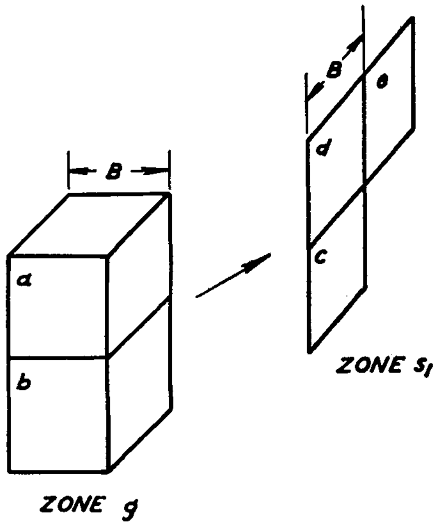

Fig. 6. Build-up of reception factors for zones from those for unit squares and cubes.

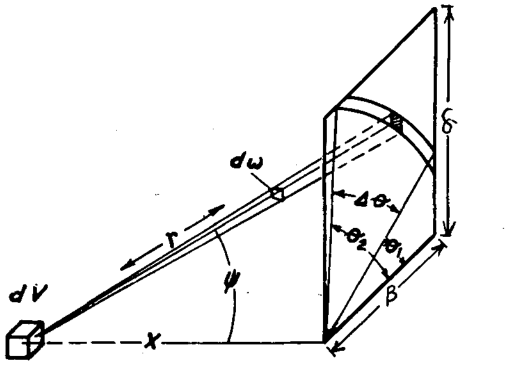

Fig. 7. Radiation from a differential-volume element to a plane.

TABLE 1. MATRIX FORMED FROM COEFFICIENTS ON W'S IN THE ENERGY BALANCES

From energy balance on Coefficients on	W_{g_1}	W_{g_2}	W_{s_1}	W_{s_2}
g_1	$(\overline{G_1G_1}-\gamma_{g_1}-\eta_{g_1}-4kaV_{g_1})$	$(\overline{G_2G_1}+\gamma_{g_2g_1}+\eta_{g_2g_1})$	$(\overline{S_1G_1}+\gamma_{s_1g_1})$	$(\overline{S_2G_1}+\gamma_{s_2g_1})$
g_2	$(\overline{G_1G_2}+\gamma_{g_1g_2}+\eta_{g_1g_2})$	$(\overline{G_2G_2}-\gamma_{g_2}-\eta_{g_2}-4kaV_{g_2})$	$(\overline{S_1G_2}+\gamma_{s_1g_2})$	$(\overline{S_2G_2}+\gamma_{s_2g_2})$
⋮	⋮	⋮	⋮	⋮
s_1	$(\overline{G_1S_1}+\gamma_{g_1s_1})$	$(\overline{G_2S_1}+\gamma_{g_2s_1})$	$(\overline{S_1S_1}-\gamma_{s_1}-\epsilon_{s_1}A_{s_1})$	$(\overline{S_2S_1})$
s_2	$(\overline{G_1S_2}+\gamma_{g_1s_2})$	$(\overline{G_2S_2}+\gamma_{g_2s_2})$	$(\overline{S_1S_2})$	$(\overline{S_2S_2}-\gamma_{s_2}-{}_{s_2}A_{s_2})$
⋮	⋮	⋮	⋮	⋮

Similar reasoning to the above leads to the recommendation that k_{avg} should be determined as the integrated mean of k along the path connecting the zone centers, from the value of k_1 at the emitting zone to k_2 at the absorbing zone, and that the interchange areas *ss*, *sg*, *gs*, and *gg* should be determined as though k were constant at k_{avg} and should then be corrected by the factors 1, k_2/k_{avg}, k_1/k_{avg}, and k_1k_2/k_{avg}^2, respectively. This procedure permits allowance for combustion products of higher concentration in the center of the combustion chamber than near the walls.

TABLE 2

LIMITING VALUES OF RECEPTION FACTORS

f_{ss}(parallel)
$= e^{-kB(r_c/B)}(\Delta z/B)^2/\pi(r_c/B)^4$
f_{ss}(perpendicular)
$= e^{-kB(r_c/B)}(\Delta x/B)(\Delta y/B)/\pi(r_c/B)^4$
$f_{gs} = e^{-kB(r_c/B)}(\Delta z/B)/4\pi(r_c/B)^3$
$f_{sg} = e^{-kB(r_c/B)}kB(\Delta z/B)/\pi(r_c/B)^3$
$f_{gg} = e^{-kB(r_c/B)}kB/4\pi(r_c/B)^2$

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NOTATION

A = area of a surface, sq. ft.
a = weighting factor, dimensionless
B = cube edge or square side, ft.
b = $\partial \ln \epsilon_g/\partial \ln T_g$, dimensionless
C_p = heat capacity, B.t.u./(lb.)(°F.)
c = exponent on temperature ratio in obtaining α_g from ϵ_g, dimensionless
D = basic determinant for system of simultaneous equations used to obtain total interchange factors
${}_{g_i}D_{s_j}$ = determinant formed by replacing the s_j column of D by a column of g_i functions
${}_{s_i}D_{s_j}$ = determinant formed by replacing the s_j column of D by a column of s_i functions
${}_{s_i}D'_{s_j}$ = minor of D formed by crossing out the s_i row and s_j column of D
d = $\partial \ln \epsilon_g/\partial \ln (PL)$, dimensionless
E = radiation-emission rate, energy per unit time
F = view factor = fraction of radiation from one surface zone intercepted by another surface zone in the presence of a nonabsorbing medium, dimensionless
$\mathfrak{F}$ = over-all interchange factor, $\mathfrak{F}_{1G}$ ($\mathfrak{F}_{12}$) = radiation reaching gas (surface A_2) due to original emission from A_1 only, but including assistance given by reflection at all surfaces, expressed as a ratio to σT_1^4, dimensionless
f_p = fraction of the energy from a differential volume element absorbed by a black plane of dimensions α and β when the volume lies on a normal through one corner of the plane and at a distance X from it, dimensionless
f = reception factor = fraction of radiation originating in one zone which reaches and is absorbed by another zone in a black-walled enclosure, dimensionless
G_m = molal flow rate per unit area, moles per hr. per sq. ft.; G_{m_n} is component of G_m normal to a given area.
$\overline{GG}$ = total interchange area between any two gas zones including reflections at all surfaces, sq. ft.; $\overline{G_1G_2}$ ($\equiv \overline{G_2G_1}$) is equal to $q_{g_1g_2}/W_{g_1}$ or $q_{g_2g_1}/W_{g_2}$
$\overline{GS}$ = total interchange area between any gas and surface zone including reflections at all surfaces, sq. ft.; $\overline{G_1S_1}$ ($\equiv \overline{S_1G_1}$) is equal to $q_{g_1s_1}/W_{g_1}$ or $q_{s_1g_1}/W_{s_1}$
g = gas zone with a numerical subscript to designate the particular zone under consideration
$\overline{gg}$ = direct interchange area between any two gas zones, sq. ft.; $\overline{g_1g_2}$ ($\equiv \overline{g_2g_1}$) is equal to $(q_{g_1g_2})_{direct}/W_{g_1}$ or $(q_{g_2g_1})_{direct}/W_{g_2}$
$\overline{gs}$ = direct interchange area between any gas and surface zone, sq. ft.; $\overline{g_1s_1}$ ($\equiv \overline{s_1g_1}$) is equal to $(q_{g_1s_1})_{direct}/W_{g_1}$ or $(q_{s_1g_1})_{direct}/W_{s_1}$
H_c = molal chemical enthalpy, B.t.u./mole
H_s = molal sensible enthalpy, B.t.u./mole
h_c = convective coefficient of heat transfer, B.t.u./(hr.) (sq. ft./°F.)
i, j = subscripts to represent any gas or surface zone
K = correction factors by which to multiply $\epsilon_{T_{avg}}$, dimensionless
k = absorption coefficient of a gas, ft.$^{-1}$
L = mean beam length for gas radiation, ft.
$\dot{M}$ = molal flow rate into system, moles/hr.

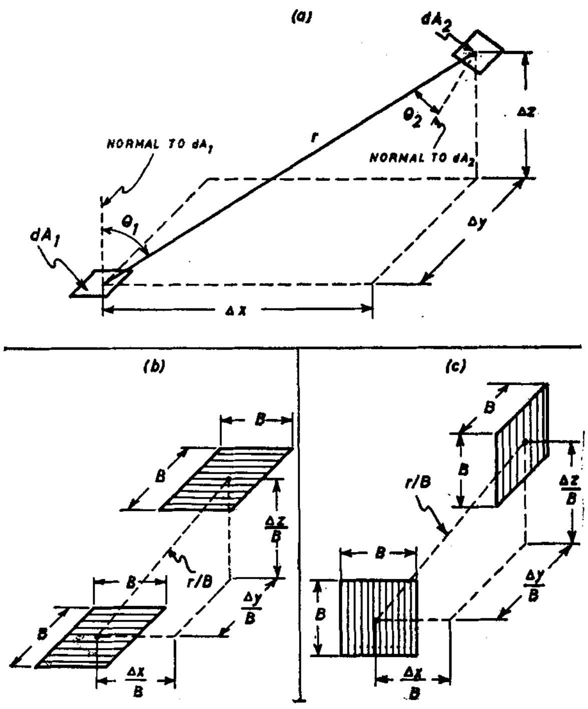

Fig. 8. Geometry of interchange between squares.

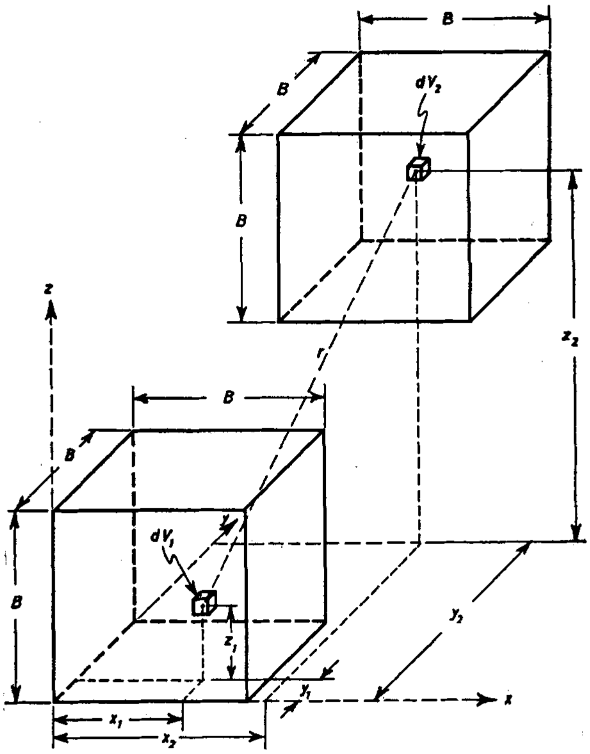

Fig. 9. Geometry of interchange between cubes.

m, n = subscripts to represent any two zones
n = a subscript to represent the number of passages through gas
P = pressure, atm.
q = heat transfer rate, B.t.u./hr.; $q_{s_i g_j}$ (or any other pair of subscripts) is the one-way energy transfer from the ith surface zone to the jth gas zone; q_s is the net heat transferred through wall s
R = relative flux density, dimensionless; ${}_{g_i}R_{s_j}$ (or any other pair of subscripts), flux density away from the jth surface zone, due solely to radiation originating within the ith gas zone, and expressed as a ratio to W_{s_i}
r = linear distance between two elements, ft.
$\overline{SG}$ = total interchange area between any surface and gas zone including reflections at all surfaces, sq. ft.; $\overline{S_1G_1}$ ($\equiv \overline{G_1S_1}$) is equal to $q_{s_1g_1}/W_{s_1}$ or $q_{g_1s_1}/W_{g_1}$
$\overline{SS}$ = total interchange area between any two surface zones including reflections at all surfaces, sq. ft.; $\overline{S_1S_2}$ ($\equiv \overline{S_2S_1}$) is equal to $q_{s_1s_2}/W_{s_1}$ or $q_{s_2s_1}/W_{s_2}$
s = surface zone with a numerical subscript to designate the particular zone under consideration
$\overline{sg}$ = direct interchange area between any surface and gas zone, sq. ft.; $\overline{s_1g_1}$ ($\equiv \overline{g_1s_1}$) is equal to $(q_{s_1g_1})_{direct}/W_{s_1}$ or $(q_{g_1s_1})_{direct}/W_{g_1}$
$\overline{ss}$ = direct interchange area between any two surface zones, sq. ft.; $\overline{s_1s_2}$ ($\equiv \overline{s_2s_1}$) is equal to $(q_{s_1s_2})_{direct}/W_{s_1}$ or $(q_{s_2s_1})_{direct}/W_{s_2}$
T = temperature, °R.; T_g, of a gas zone; T_s, of a surface zone; T_0, of system surroundings; T_{avg}, arithmetic mean of two temperatures
U_s = over-all coefficient of heat transfer, inner surface of wall s to surroundings at T_0, B.t.u./(hr.)(sq. ft./°R.)
u = number of zones of known temperature, dimensionless
V = gas volume, cu. ft.
W = rate of emission from unit area of black body at temperature indicated by subscript, B.t.u./(hr.)(sq. ft.)
$\dot{w}$ = weight flow rate, lb./hr.
X = normal distance between a differential radiating volume element and a plane, ft.
x, y, z = linear dimensions along the three coordinate axes, ft.

Greek Letters

α = absorptivity, dimensionless
γ = convective flux coefficient, sq. ft.
Δ = a difference
ϵ = emissivity, dimensionless
η = enthalpy flux coefficient, sq. ft.
θ = angle, radius
μ = refractive index, dimensionless
ρ = reflectivity, dimensionless
σ = Stefan-Boltzmann constant = 0.1713×10^{-8} B.t.u./(hr.)(sq. ft./°R.^{4})
τ = gas transmissivity, dimensionless
ϕ = escape factor for radiation from a cube of gas (ratio of energy leaving boundaries to energy originating within volume), dimensionless
ψ = angle, radians
ω = solid angle, steradians

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